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ON THE PROPERTIES AND CALCULATION OF CERTAIN INTEGRALS FROM THE STATISTICAL THEORY OF RESIDUAL CURRENTS IN TIDAL AREAS

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On the properties and calculation of certain integrals from the statistical theory of residual currents in tidal areas

by

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ABSTRACT

Asymptotic approximations are given, together with algorithms for numerical evaluations, for integrals occurring in a mathematical model of residual circulation by tidal currents. The integrals contain functions of the hypergeometric type.

KEYWORDS & PHRASES: *asymptotic expansion, Bessel function, hypergeometric function, numerical quadrature, tidal currents*

1. INTRODUCTION

In ZIMMERMAN (1978) aspects of residual circulation by tidal currents are discussed. In the mathematical model integrals are encountered, which are too complicated for straightforward computations. In this note we describe the methods for obtaining numerical values of the integrals. Besides, some analytical expansions will be considered.

The vorticity of a two-dimensional velocity field $\vec{u}(\vec{x},t)$ is defined by $\eta(\vec{x},t) = \nabla \times \vec{u}(\vec{x},t)$ (in two-dimensional velocity fields η can be treated as a scalar as it has only a component perpendicular to the plain of motion). The vorticity η is subsequently written as a spatial Fourier integral. The individual Fourier components are then described by

$$\frac{d\eta(\vec{k},t)}{dt} - [ik_1 \sin t + b(k)]\,\eta(\vec{k},t) = a(\vec{k})\sin t\,, \tag{1.1}$$

where $\vec{k}$ is the wave number of the Fourier component and $a(\vec{k})$ and $b(k)$ are time-independent parameters, $\vec{k} = (k_1,k_2)$, $k = (k_1^2+k_2^2)^{\frac{1}{2}}$. The term $ik_1 \sin t$ is a time dependent damping term produced by the oscillatory tidal velocity constant $\times \sin t$. The driving term $a \sin t$ is in the same manner time dependent. The interaction of a driving term with a damping term of the same frequency produces a residual effect.

The solution of (1.1) is easily obtained:

$$\eta(t) = a\, e^{-ik_1 \cos t + bt} \int^{t} \sin\tau\; e^{ik_1 \cos\tau - b\tau}\, d\tau. \tag{1.2}$$

Not this general solution is needed, but the time independent (residual) part of the solution; i.e., the stationary non-periodic component of the Fourier expansion of $\eta(t)$. This is obtained (Zimmerman) by substituting the well-known expansion from Bessel function theory

$$e^{ik_1 \cos\tau} = \sum_{m=-\infty}^{\infty} i^m e^{im\tau} J_m(k_1) \tag{1.3}$$

(and a similar expansion for $\exp(-ik_1 \cos t)$) and computing the time independent term in the resulting double series. Denoting the result again by η, depending on $\vec{k}$, it reads as

$$(1.4) \qquad \eta(\vec{k}) = \frac{ia}{k_1} \sum_{n=-\infty}^{\infty} \frac{n^2}{n^2+b^2} J_n^2(k_1).$$

The solution of (1.1) is now treated statistically, i.e., the parameter $a(\vec{k})$ is a random function of $\vec{k}$, which therefore randomizes $\eta(\vec{k})$. The quantity of primary interest is the mean square of $\eta(\vec{k})$ often called the "enstrophy" (the residual enstrophy here). It can be shown that $\langle\eta^2\rangle$ is proportional to

$$(1.5) \qquad g_1(\lambda) = 4\lambda^{-6}\pi^{-1} \int_0^{\infty} k^5 \, G(k) e^{-k^2\lambda^{-2}} dk, \qquad \lambda > 0.$$

A related quantity is the mean square velocity, proportional to

$$(1.6) \qquad g_2(\lambda) = 4\lambda^{-4}\pi^{-1} \int_0^{\infty} k^3 \, G(k) e^{-k^2\lambda^{-2}} dk,$$

where

$$(1.7) \qquad G(k) = \int_0^{\frac{1}{2}\pi} F(k \cos\theta) d\theta, \quad F(k_1) = k_1^{-2}\left[\sum_{n=-\infty}^{\infty} \frac{n^2}{n^2+b^2} J_n^2(k_1)\right]^2.$$

In (1.1) the parameter $b(k)$ controls the dissipation of vorticity by frictional effects. In fact, two different processes contribute to the dissipation, viz. bottom friction and horizontal vorticity diffusion. In the parameter b they appear as a sum

$$(1.8) \qquad b(k) = 1+\tau k^2.$$

The first term stands for the bottom friction, the second one for horizontal diffusion. In the latter part the parameter τ can be chosen freely. In this note the cases $\tau = 0$ and $\tau > 0$ are investigated.

In the following sections we give information on the asymptotic behaviour of $g_1(\lambda)$ and $g_2(\lambda)$, (for $\lambda \to 0$ and $\lambda \to \infty$) and on the numerical evaluation of these functions.

2. ANALYTICAL EXPANSIONS

First we evaluate the series in (1.4) and (1.7) in closed form. The result is

$$(2.1) \qquad \sum_{n=-\infty}^{\infty} \frac{n^2}{n^2+b^2} J_n^2(x) = 1 - \frac{\pi b}{\sinh \pi b} J_{-ib}(x) J_{ib}(x).$$

For a proof of this relation we first mention the well-known property for the Bessel functions

$$(2.2) \qquad \sum_{n=-\infty}^{\infty} J_n^2(x) = 1,$$

which is a consequence of Parseval's formula for the Fourier series (1.3). Hence, it remains to prove

$$(2.3) \qquad \sum_{n=-\infty}^{\infty} \frac{b^2}{n^2+b^2} J_n^2(x) = \frac{\pi b}{\sinh \pi b} J_{-ib}(x) J_{ib}(x).$$

Consider the integral

$$(2.4) \qquad \frac{1}{2\pi i} \int_C \frac{J_{-\nu}(x) J_{+\nu}(x)}{\nu^2+b^2} \frac{d\nu}{\sin \nu\pi},$$

where C encloses the poles $\nu = 1,2,\ldots$ of $\sin \nu\pi$ so that it cuts the real ν-axis in $\nu = 0$ perpendicularly. We suppose temporarily that b is a complex number such that $|\arg ib| < \pi/2$, and we suppose that C does not enclose the poles $\nu = \pm ib$. Deforming C into the axis $\mathrm{Re}\,\nu = 0$, we pass the pole at $\nu = ib$ and the resulting integral vanishes since the integrand of (2.4) is an odd function of ν. By using the residues we arrive at (2.3). Using the principle of analytic continuation, we infer that (2.3) is valid for all $b \in \mathbb{C}$ except for the points $\pm i, \pm 2i, \ldots$.

As a consequence, the function F of (1.7) can be written as

$$(2.5) \qquad F(k_1) = k_1^{-2}[1 - \frac{\pi b}{\sinh \pi b} J_{-ib}(k_1) J_{ib}(k_1)]^2,$$

or in terms of hypergeometric functions

$$(2.6) \qquad F(k_1) = k_1^{-2}[1 - {}_1F_2(\tfrac{1}{2}; 1+ib, 1-ib; -k_1^2)]^2,$$

where the hypergeometric function ${}_1F_2$ is defined as

$$(2.7) \qquad {}_1F_2(\alpha;\beta,\gamma;x) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+n)\Gamma(\gamma+n)} \frac{x^n}{n!} .$$

The expansion of $F(k_1)$ in powers of k_1 (considering temporarily b independent of k_1) is thus given as follows

$$F(k_1) = k_1^2 \Big[\sum_{m=0}^{\infty} a_m k_1^{2m} \Big]^2 = \sum_{m=0}^{\infty} b_m k_1^{2m+2}, \tag{2.8}$$

with

$$a_m = \frac{(-1)^m}{(m+1)!} \frac{\Gamma(m+3/2)}{\Gamma(1/2)} \frac{\Gamma(1+ib)}{\Gamma(m+2+ib)} \frac{\Gamma(1-ib)}{\Gamma(m+2-ib)}, \quad m \geq 0;$$

$$\begin{aligned} a_0 &= \frac{1/2}{1+b^2}, \quad a_1 = \frac{-3/4}{(1+b^2)(4+b^2)2!}, \\ a_m &= -\frac{(m+1/2)}{(m+1)} \frac{a_{m-1}}{(m+1)^2+b^2}, \quad m \geq 1; \end{aligned} \tag{2.9}$$

$$b_m = \sum_{j=0}^{m} a_j a_{m-j}, \quad m \geq 0.$$

Furthermore we can expand the function G of (1.7) as well. The series in (2.8) is substituted in the first equation of (1.7) (owing to the convergence properties the integral may be integrated term by term) and the result is

$$\begin{aligned} G(k) &= \sum_{m=0}^{\infty} c_m k^{2m+2}, \\ c_m &= b_m \int_0^{\frac{1}{2}\pi} \cos^{2m+2}\phi \, d\phi = \tfrac{1}{2}\sqrt{\pi} \frac{\Gamma(m+3/2)}{(m+1)!} b_m, \quad m \geq 0. \end{aligned} \tag{2.10}$$

The first few c_m are

$$\begin{aligned} c_0 &= \frac{\pi}{16} (1+b^2)^{-2} \\ c_1 &= -\frac{9\pi}{128} (1+b^2)^{-2}(4+b^2)^{-1} \\ c_2 &= \frac{5\pi}{2048} (161+29b^2)(1+b^2)^{-2}(4+b^2)^{-2}(9+b^2)^{-1}. \end{aligned} \tag{2.11}$$

REMARK. From the convergence properties of the series in (2.7), it follows that the power series in (2.8) and (2.10) converge for all finite values of k_1 and k, respectively. They give "good" representations of F and G for small values of k_1 and k. As indicated in (1.8), the parameter b depends on k. The character of the series in (2.8) and (2.10) is completely different for the cases $\tau = 0$ and $\tau > 0$. As mentioned above, for $\tau = 0$ they give good representations for F and G for small k_1 and k. However, for large values of these parameters and $\tau = 0$, the series are useless for describing the behaviour of F and G. For $\tau > 0$, on the other hand, we have

$$c_0 = \mathcal{O}(k^{-8}),\ c_1 = \mathcal{O}(k^{-12}),\ c_2 = \mathcal{O}(k^{-16}),\ k \to \infty, \tag{2.12}$$

and in general $c_m = \mathcal{O}(k^{-8-4m})$, $k \to \infty$. Hence, for $\tau > 0$, the terms $c_m\,k^{2m+2}$ in the series (2.10) behave for $k \to \infty$ as $\mathcal{O}(k^{-6-2m})$. Therefore it is concluded, that, if $\tau > 0$, the series in (2.10) gives a good representation of G for both cases $k \to 0$ and $k \to \infty$. In a first approximation we have

$$G(k) \sim c_0 k^2 = \begin{cases} \mathcal{O}(k^2) & k \to 0 \quad \tau \geq 0 \\ \mathcal{O}(k^{-6}) & k \to \infty \quad \tau > 0 \end{cases} \tag{2.13}$$

In the applications τ is rather small. Therefore for numerical evaluations (2.10) is not very useful for intermediate values of k.

2.1. Asymptotic expansion of $g_i(\lambda)$ for $\lambda \to 0$

In order to consider both g_i we introduce the function

$$H_\alpha(\lambda) = 4\lambda^{-4}\pi^{-1} \int_0^\infty k^{2\alpha+1}\ G(k)e^{-k^2\lambda^{-2}} dk,\ \lambda > 0, \tag{2.14}$$

where G is the same as in (1.5), (1.6) and (1.7). From (1.5) and (1.6) it follows that

$$g_1(\lambda) = \lambda^{-2}\ H_2(\lambda),\ g_2(\lambda) = H_1(\lambda). \tag{2.15}$$

For obtaining the asymptotic behaviour of integrals of the type

$$\int_0^\infty e^{-st}\ f(t)dt$$

for $s \to \infty$, the asymptotic behaviour of f for $t > 0$ is substituted. (Watson's lemma; see any book on asymptotic analysis, for instance BLEISTEIN & HANDELSMAN (1975)). From (2.11) and (2.13) we know that

$$G(k) \sim \frac{\pi}{64}\,k^2,\ k \to 0,\ \tau \geq 0.$$

Hence for the behaviour of $H_\alpha(\lambda)$ for $\lambda \to 0$ we obtain

$$(2.16)\qquad H_\alpha(\lambda) \sim \frac{1}{16}\lambda^{-4}\int_0^\infty k^{2\alpha+3}e^{-k^2\lambda^{-2}}dk = \frac{1}{32}\Gamma(\alpha+2)\lambda^{2\alpha}.$$

Using (2.15) we obtain for the functions g_1 and g_2

$$(2.17)\qquad g_1(\lambda) \sim \frac{3}{16}\lambda^2,\quad g_2(\lambda) \sim \frac{1}{16}\lambda^2,\ \lambda \to 0,\ \tau \geq 0.$$

The results in (2.16) and (2.17) are first approximations. For numerical applications we compute some higher order terms. Thus the role of the parameter τ is better understood in the limit case $\lambda \to 0$. Substitution of the complete series (2.10) for G in (2.14) yields (we recall that c_m depends on k, hence we write $c_m(k)$)

$$H_\alpha(\lambda) \sim 4\pi^{-1}\lambda^{-4}\sum_{m=0}^{\infty} A_m,$$

with

$$A_m = \int_0^\infty c_m(k)k^{2\alpha+2m+3}e^{-k^2\lambda^{-2}}dk.$$

Subsequently each $c_m(k)$ is expanded in powers of k and this expansion is substituted in the above integral for A_m. For the first few $c_m(k)$ the expansions are (see (2.11) and (1.8))

$$c_0(k) = \frac{\pi}{16}[1+(1+\tau k^2)]^{-2} = \frac{\pi}{16}(\tfrac{1}{4} - \tfrac{1}{2}\tau k^2 + \tfrac{1}{2}\tau^2 k^4 \ldots),$$

$$c_1(k) = -\frac{9\pi}{128}\left(\frac{1}{20} - \frac{3}{25}\tau k^2 + \ldots\right)$$

$$c_2(k) = \frac{5\pi}{2048}\left(\frac{19}{100} + \ldots\right).$$

With the computed numbers in these expansions we obtain by substituting these results and by collecting equal powers in λ

$$(2.18)\qquad H_\alpha(\lambda) \sim \frac{2}{\pi}\lambda^{2\alpha}\Gamma(\alpha+2)(h_0+h_1\lambda^2+h_2\lambda^4+\ldots),$$

with

$$h_0 = \frac{\pi}{64},$$

$$h_1 = -\pi(\alpha+2)\left(\frac{9}{2560} + \frac{\tau}{32}\right),$$

$$h_2 = \pi(\alpha+2)(\alpha+3)\left(\frac{19}{40960} + \frac{27\tau}{3200} + \frac{\tau^2}{32}\right).$$

By using (2.15) we obtain for g_1 and g_2

$$g_1(\lambda) \sim \frac{3}{16}\,\lambda^2[1-\,(9/10+8\tau)\lambda^2 + (19/32+108\tau/10+40\tau^2)\lambda^4+\ldots],$$

(2.19)

$$g_2(\lambda) \sim \frac{1}{16}\,\lambda^2[1-\,(27/40+6\tau)\lambda^2 + (57/160+162\tau/25+24\tau^2)\lambda^4+\ldots].$$

These expansions are truncated after the term with λ^4. The error is then $\mathcal{O}(\lambda^6)$. From numerical experiments it follows indeed that with the given coefficients in (2.19) a relative accuracy can be obtained smaller than 10^{-3}, if we take $\lambda^2 \le 10^{-1}$.

2.2. Asymptotic expansion of $g_i(\lambda)$ for $\lambda \to \infty$

It is necessary to distinguish between the two cases $\tau = 0$ and $\tau > 0$. The functions have, as will be shown, a quite different behaviour in these cases.

2.2.1. The case $\tau > 0$

The asymptotic behaviour of the integral in (2.14) for $\lambda \to \infty$ is obtained by replacing the exponential function $\exp(-k^2\lambda^{-2})$ by its limiting value 7, if the resulting integral $\int k^{2\alpha+1}\, G(k)dk$ converges. From (2.13) it follows that $G(k) = \mathcal{O}(k^{-6})$, $k \to \infty$. Hence, the resulting integral converges if $2\alpha+1-6 < -1$, or $\alpha < 2$. From (2.15) it thus follows that

$$(2.20)\qquad g_2(\lambda) \sim 4\lambda^{-4}\pi^{-1}\int_0^\infty k^3\, G(k)dk, \quad \lambda \to \infty.$$

For g_1 this rule does not apply: we need some more information on G. For $\alpha = 2$, the above mentioned resulting integral does not converge owing to the behaviour of the integrand for $k \to \infty$. Substituting (2.13) in (1.5) we obtain (by using (2.11))

$$g_1(\lambda) \sim \tfrac{1}{4}\lambda^{-6}\int_0^\infty k^7 e^{-k^2\lambda^{-2}}\,\frac{dk}{[1+(1+\tau k^2)^2]^2}, \quad \lambda \to \infty.$$

This integral can be expressed in terms of hypergeometric functions, but this is not an important aspect. We are interested in the asymptotic behaviour for $\lambda \to \infty$. By examining further the behaviour of the above integrand it appears that we have

$$\int_0^\infty k^5 G(k) e^{-k^2\lambda^{-2}} dk = \int_c^\infty k^{-1} e^{-k^2\lambda^{-2}} dk + \mathcal{O}(1), \quad \lambda \to \infty$$

with c any positive number (not depending on λ). This is written as

$$\int_0^\infty k^5 G(k) e^{-k^2\lambda^{-2}} dk = \tfrac{1}{2}\int_{c^2/\lambda^2}^\infty e^{-t} t^{-1} dt + \mathcal{O}(1) =$$

$$= \tfrac{1}{2}\int_{c^2/\lambda^2}^\infty t^{-1} dt + \mathcal{O}(1) = \ln\lambda + \mathcal{O}(1), \ \lambda \to \infty,$$

which gives

(2.2.1) $\quad g_1(\lambda) \sim \tfrac{1}{4}\tau^{-4}\lambda^{-6}\ln\lambda, \quad \lambda \to \infty$

of course under the restriction $\tau \neq 0$.

2.2.1. The case $\tau = 0$

In this case (2.13) is not appropriate for describing the behaviour of G for $k \to \infty$. It is better to use a different representation of the function $H_\alpha(\lambda)$. First we remark that (2.14) (with G given in (1.7)) is written in polar coordinate form. Originally, in the physical problem (see Zimmerman's paper), the integration is carried out with respect to the wave number parameters

$$k_1 = k\cos\theta, \quad k_2 = k\sin\theta.$$

Integrating with respect to (k_1,k_2), we arrive at the representation

(2.2.2) $$H_\alpha(\lambda) = \pi^{-1}\lambda^{-4}\int_{-\infty}^\infty\int_{-\infty}^\infty (k_1^2+k_2^2)^\alpha e^{-(k_1^2+k_2^2)\lambda^{-2}} F(k_1)dk_1dk_2$$

where F is given in (1.7) or in (2.5).

Now, if α is an integer number, the term $(k_1^2+k_2^2)^\alpha$ polynomial in k_2, and we integrate first with respect to k_2. Then $H_\alpha(\lambda)$ can be written as

$$H_1(\lambda) = \pi^{-\frac{1}{2}}\lambda^{-3}[A_2(\lambda) + \tfrac{1}{2}\lambda^2 A_0(\lambda)],$$

(2.23)

$$H_2(\lambda) = \pi^{-\frac{1}{2}}\lambda^{-3}[A_4(\lambda) + \lambda^2 A_2(\lambda) + \tfrac{3}{4}\lambda^4 A_0(\lambda)],$$

with

(2.24) $$A_{2j}(\lambda) = \int_{-\infty}^{\infty} k_1^{2j}\, F(k_1)e^{-k_1^2\lambda^{-2}}\, dk_1 \qquad j = 0,1,2,.$$

$$= 2\, \lambda^{2j+1} \int_0^{\infty} e^{-t^2} t^{2j}\, F(\lambda t)dt.$$

For the asymptotic expansion of $A_{2j}(\lambda)$ $(\lambda \to \infty)$ we need the asymptotic behaviour of $F(t)$ for $t \to \infty$, where F is given in (2.5) with $b = 1$ (for $\tau = 0$). Hence, we need the asymptotic expansion of the Bessel functions $J_{\pm i}(t)$ for $t \to \infty$.

From the literature (see, for instance, WATSON (1945)) we know

(2.25) $$J_\nu(x) = (2/\pi x)^{\frac{1}{2}}\{P(\nu,x)\cos\chi - Q(\nu,x)\sin\chi\},$$

where, for $x \to \infty$, P and Q can be represented by asymptotic power series

(2.26) $$P(\nu,x) \sim \sum_{m=0}^{\infty} (-1)^m(\nu,2m)(2x)^{-2m},\quad Q(\nu,x) \sim \sum_{m=0}^{\infty} (-1)^m(\nu,2m+1)(2x)^{-m-1},$$

with

(2.27) $$(\nu,m) = \frac{\Gamma(\frac{1}{2}+\nu+m)}{m!\Gamma(\frac{1}{2}+\nu-m)} = \frac{2^{-2m}}{m!}\{(4\nu^2-1)(4\nu^2-3^2)\ldots[4\nu^2-(2m-1)^2]\};$$

in (2.25) χ is given by

(2.28) $$\chi = x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi.$$

With (2.25) we obtain

(2.29) $$J_{-i}(x)J_i(x) \sim \frac{1}{\pi x}[P^2(\cosh\pi + \sin 2x) + Q^2(\cosh\pi - \sin 2x) + 2PQ\cos 2x],$$

where P and Q stand for $P(i,x)$ and $Q(i,x)$. Hence

(2.30) $$F(x) \sim \frac{1}{x^2}[1 + \frac{f_1}{x} + \frac{f_2}{x^2} + \frac{f_2}{x^3} + \ldots],$$

where f_i may contain circular functions. By formal manipulations we obtain the first few coefficients:

$$f_1 = 2\,\frac{\cosh\pi + \sin 2x}{\sinh\pi},$$

(2.31) $$f_2 = \frac{2\cosh^2\pi + \cosh\pi\sin 2x+1 - \cos 4x + 5\sinh\pi\cos 2x}{2\sinh^2\pi}$$

$$f_3 = \frac{5}{4}\,(1-2\cos 2x)\coth\pi + \frac{45}{16}\,\frac{\sin 2x}{\sinh\pi} - \frac{5}{4}\,\frac{\sin 4x}{\sinh\pi}\,.$$

After these preparations we give the asymptotic expansion of $A_{2j}(\lambda)$ $(\lambda\to\infty)$ of (2.24). Applying the Mellin-transform technique (see BLEISTEIN & HANDELSMAN (1975, ch.4)) we obtain

(2.32) $$A_{2j}(\lambda) = \frac{\lambda^{2j+1}}{2\pi i}\int_{-i\infty}^{i\infty} \lambda^{-z}\ \Gamma(j+\tfrac{1}{2}-z/2)\ M[F;z]dz,$$

with

$$M[F;z] = \int_0^\infty t^{z-1}\ F(t)dt.$$

By moving the contour of integration in (2.32) we obtain an expansion of $A_{2j}(\lambda)$ by calculating residues at the poles of $\Gamma(j+1/2-z/2)$ (at $z = 2j+1+2k$, $k = 0,1,2,\ldots$) and at the poles of $M[F;z]$ (at $z = 2,3,4,\ldots$, as follows from the asymptotic behaviour of F given in (2.28)). Hence, for $j = 0$ the first pole lies at $z = 1$, while for $j = 1,2$ it lies at $z = 2$. This gives the following results (for details the reader is referred to the literature)

$$A_0(\lambda) \sim 2M[F;1],$$

$$A_2(\lambda) \sim \lambda\sqrt{\pi},$$

$$A_4(\lambda) \sim \tfrac{1}{2}\sqrt{\pi}\,\lambda^3,$$

and combining these results in (2.23) we obtain for $\lambda\to\infty$

$$H_1(\lambda) = \pi^{-\frac{1}{2}}M[F;1]\lambda^{-1} + \mathcal{O}(\lambda^{-2}),$$

(2.33)

$$H_2(\lambda) = \pi^{-\frac{1}{2}}\,\frac{3}{2}\,M[F;1] + \mathcal{O}(1).$$

For g_1 and g_2 we obtain with these formulas for $\lambda\to\infty$ (and $\tau = 0$)

$$g_1(\lambda) = \pi^{-\frac{1}{2}}\,\frac{3}{2}\,M[F;1]\lambda^{-1} + \mathcal{O}(\lambda^{-2}),$$

(2.34)

$$g_2(\lambda) = \pi^{-\frac{1}{2}}\,M[F;1]\lambda^{-1} + \mathcal{O}(\lambda^{-2}),\quad \text{for } \lambda\to\infty \text{ (and } \tau = 0).$$

3. THE NUMERICAL COMPUTATION OF $H_\alpha(\lambda)$

The function $H_\alpha(\lambda)$ defined in (2.14) will be computed by using a numerical quadrature process, except for small values of λ. In the quadrature process we need the function $G(k)$, which does not depend on λ. In this section we pay attention to this problem. We try to compute G with a relative accuracy of about 10^{-5}. In the final computation of $H_\alpha(\lambda)$ we try to obtain a relative accuracy of 10^{-3}.

For small values of k we use the expansion given in (2.10), for both cases $\tau = 0$ and $\tau > 0$. It appears that the series is alternating and that for large or intermediate values of k cancellation of digits may occur. A satisfactory bound is $k \leq 6.5$. For $k = 6.5$ about 20 terms in the series are needed. Numerical experiments learned that the cancellation is still under control for this value.

For $k \geq 6.5$ we use the representation of G given in (1.7). Now we need for numerical quadrature the function $F(k_1)$.

3.1 Computation of F

For small values of its argument, F can be computed by using (2.8). Again, we use as a bound $k_1 \leq 6.5$. For larger values it is necessarily to distinguish between the cases $\tau = 0$ and $\tau > 0$.

3.1.1. Computation of F for $\tau = 0$, $k_1 \geq 6.5$

In this event we use (2.5) (with $b = 1$) and the asymptotic expansions mentioned in (2.25), (2.26) and (2.29). By computing the first coefficients in the series we obtain

$$P(i,x) \simeq 1 - 0.507813\, x^{-2} + 1.016286\, x^{-4} - 5.623980\, x^{-6} + 62.161631\, x^{-8},$$

$$Q(i,x) \simeq -0.625\, x^{-1}(1 - 0.981771\, x^{-2} + 3.455373\, x^{-4} - 27.798528\, x^{-6} + 404.773839\, x^{-8}).$$

With these approximation the desired accuracy in F can be realized.

3.1.2. Computation of F for $\tau > 0$, $k_1 \geq 6.5$

In this event the asymptotic approximations used above lose their usefulness, since the order of the Bessel functions heavily depends on k. Direct computation of the Bessel functions in (2.5) is rather complicated owing to this. Therefore we propose to use the series (1.7). This asks for the computation of a sequence of Bessel functions $J_1, J_2, \ldots$, but algorithms for this problem are readily available. Because of the positivity of the terms in the series

$$(3.1) \qquad F(k_1) = 4k_1^{-2} \sum_{n=1}^{\infty} \frac{n^2}{n^2+b^2} J_n^2(k_1), \quad b = 1+\tau k^2,$$

we have no stability problems in summing the series, when the Bessel functions are available.

The first problem is to determine the integer N such that the first N terms in (3.1) give an accuracy of 5 significant figures. If $n > k_1$, the terms decrease very fast if n increases. It is sufficient to choose N such that

$$(3.2) \qquad J_N^2(k_1) < 10^{-5}.$$

The inversion of the equation $J_\nu^2(x) = 10^{-5}$ with $\nu > x$, i.e., the computation of ν if x is given, can be established (in an approximative sense) by using Debye's approximation

$$J_\nu(\nu/\cosh\alpha) \sim e^{\nu(\tanh\alpha-\alpha)} (2\pi\nu\tanh\alpha)^{-\frac{1}{2}}, \quad \nu \to \infty.$$

(see WATSON (1944, p.243)). We try to solve the equation in ν:
$\exp[2\nu(\tanh\alpha-\alpha)] = \exp[2x(\sinh\alpha-\alpha\cosh\alpha)] = 10^{-5}$, with $x = \nu/\cosh\alpha$, or equivalently, the equation in α:

$$\alpha\cosh\alpha - \sinh\alpha = p, \quad p = 5.76/x.$$

When α is computed, we have $\nu = x\cosh\alpha$. Since $p < 1$ (because of $x \geq 6.5$, x plays the role of k_1), α is rather small. We invert therefore the above equation with Taylor series. This results in an approximation

$$\alpha \simeq y(1 + a_1y^2 + a_2y^4 + a_3y^6), \quad y = (3p)^{1/3},$$

which can be substituted in $\nu = x \cosh \alpha$, yielding (after computation of the numbers a_1, a_2, a_3)

$$\nu \simeq x\left(1 + \frac{1}{2}y^2 + \frac{1}{120}y^4 - \frac{1}{2800}y^6\right),$$

or in terms of N, k_1

$$N \simeq k_1\left(1 + \frac{1}{2}y^2 + \frac{1}{120}y^4 - \frac{1}{2800}y^6\right), \qquad y = \left(\frac{12.27}{k_1}\right)^{1/3} .$$

With this value of N we obtain the desired accuracy in the approximation

$$F(k_1) \simeq 4k_1^{-2} \sum_{n=1}^{N} \frac{n^2}{n^2+b^2} J_n^2(k_1).$$

For the computation of the Bessel functions we use the NUMAL procedure BESS J, code 35162. However, it is designed for an accuracy of about 10^{-14}. Therefore we adapted it for the present situation. The algorithm is given in the text of the ALGOL 60 program in section 4.

Several tests are carried out for checking the reliability of the algorithms. Especially we computed F at the right and at the left of the "turning point" $k_1 = 6.5$. The ultimate tests were satisfactory.

3.2. The computation of G(k)

As mentioned in the introductory part of this section, we used the expansion of (2.10) for $0 \le k_1 \le 6.5$. For $k_1 \ge 6.5$ the integration in (1.7) is carried out mechanically, by using the trapezoidal rule. For periodic analytic functions this rule is rather efficient. See for more details, for instance, TEMME (1977). As outlined in the above subsection the values of $F(k \cos \Theta)$ are obtained from expansion (2.8) or from asymptotic expansions for the Bessel functions in (2.5) (if $\tau = 0$) or the series in (1.7) (if $\tau > 0$).

3.3. The computation of $H_\alpha(\lambda)$

For $0 < \lambda < 10^{-1}$ we used the asymtptotic expansion (2.18) up to and including the term wit λ^4. An estimation of the termination error is not available (i.e., different from $\mathcal{O}(\lambda^6)$) but we compared the series with the quadrature method discussed further on. The results where comparable within the desired relative accuracy of 10^{-3}. Even for $\lambda = 1/\sqrt{10}$ the asymptotic expansion gives the accuracy.

For $\lambda \geq 10^{-1}$ we used a trapezoidal rule. In order to obtain a representation of $H_\alpha(\lambda)$ that is suited for this rule we transformed the variable of integration by writing

$$\text{(3.3)} \qquad k = e^{x-e^{-x}}, \quad (k \in [0,\infty), \quad x \in (-\infty,\infty))$$

giving

$$H_\alpha(\lambda) = 4\pi^{-1}\lambda^{-4} \int_{-\infty}^{\infty} k^{2\alpha+2} \, (1+e^{-x}) e^{-\lambda^{-2}k^2} G(k)dx,$$

with k given in (3.3). For the values of α and λ relevant for the numerical process ($10^{-2} \leq \lambda \leq 10^2$, $\alpha = 1,2$) we approximated the above integral by choosing as end points of integration -1 and 6.5. On the x-interval [-1,6.5] we applied the trapezoidal rule. Again, the above integral is well suited for this rule (TEMME (1977)). We pretabulated the function G(k) (as function of x) in equidistant points of the x-interval [-1,6.5] (including the end points) with step size $h = 7.5\times2^{-6} = 0.1171875$. So the computation of G was done once and for all (for $\tau = 0, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}$) independent of the λ-value and the α-value. This resulted in a reliable and efficient algorithm for $H_\alpha(\lambda)$. The tables show the results of the numerical computations, which were done for $y = {}^{10}\log \lambda = -2(.1)2$.

<u>REMARK</u>. For $\tau = 0$ we might have used the asymptotic expansion for $\lambda \to \infty$ as considered in subsection 2.2.2, see (2.34). From this formulas, it follow that for $\lambda \to \infty$, $\tau = 0$ we have approximately $g_1(\lambda) = \frac{3}{2}\, g_2(\lambda)$. The table confirms this relation. (For $\lambda = 100$, we compute from the table $g_1(\lambda)/g_2(\lambda) = 1.49774\ldots$.) The computation of higher order terms, however, is rather intricate, so we abandoned this method for numerical computations.

TAU = 0.0"+00

Y	LABDA	G1	G2
-2.0	.100"-01	.167"-04	.625"-05
-1.9	.126"-01	.297"-04	.990"-05
-1.8	.158"-01	.471"-04	.157"-04
-1.7	.200"-01	.746"-04	.249"-04
-1.6	.251"-01	.118"-03	.394"-04
-1.5	.316"-01	.187"-03	.625"-04
-1.4	.398"-01	.297"-03	.989"-04
-1.3	.501"-01	.470"-03	.157"-03
-1.2	.631"-01	.744"-03	.248"-03
-1.1	.794"-01	.118"-02	.393"-03
-1.0	.100"+00	.186"-02	.621"-03
-0.9	.126"+00	.293"-02	.980"-03
-0.8	.158"+00	.461"-02	.154"-02
-0.7	.200"+00	.720"-02	.242"-02
-0.6	.251"+00	.112"-01	.378"-02
-0.5	.316"+00	.172"-01	.585"-02
-0.4	.398"+00	.259"-01	.893"-02
-0.3	.501"+00	.380"-01	.134"-01
-0.2	.631"+00	.536"-01	.194"-01
-0.1	.794"+00	.716"-01	.269"-01
0.0	.100"+01	.890"-01	.354"-01
0.1	.126"+01	.102"+00	.432"-01
0.2	.158"+01	.106"+00	.487"-01
0.3	.200"+01	.102"+00	.508"-01
0.4	.251"+01	.930"-01	.495"-01
0.5	.316"+01	.816"-01	.457"-01
0.6	.398"+01	.697"-01	.407"-01
0.7	.501"+01	.583"-01	.352"-01
0.8	.631"+01	.481"-01	.298"-01
0.9	.794"+01	.393"-01	.248"-01
1.0	.100"+02	.318"-01	.203"-01
1.1	.126"+02	.256"-01	.166"-01
1.2	.158"+02	.205"-01	.134"-01
1.3	.200"+02	.164"-01	.108"-01
1.4	.251"+02	.131"-01	.863"-02
1.5	.316"+02	.104"-01	.689"-02
1.6	.398"+02	.829"-02	.550"-02
1.7	.501"+02	.659"-02	.438"-02
1.8	.631"+02	.524"-02	.349"-02
1.9	.794"+02	.417"-02	.277"-02
2.0	.100"+03	.331"-02	.221"-02

TAU = 1.0"-05

Y	LABDA	G1	G2
-2.0	.100"-01	.187"-04	.625"-05
-1.9	.126"-01	.297"-04	.990"-05
-1.8	.158"-01	.471"-04	.157"-04
-1.7	.200"-01	.746"-04	.249"-04
-1.6	.251"-01	.118"-03	.394"-04
-1.5	.316"-01	.187"-03	.625"-04
-1.4	.398"-01	.297"-03	.989"-04
-1.3	.501"-01	.470"-03	.157"-03
-1.2	.631"-01	.744"-03	.248"-03
-1.1	.794"-01	.118"-02	.393"-03
-1.0	.100"+00	.186"-02	.621"-03
-0.9	.126"+00	.293"-02	.980"-03
-0.8	.158"+00	.461"-02	.154"-02
-0.7	.200"+00	.720"-02	.242"-02
-0.6	.251"+00	.112"-01	.378"-02
-0.5	.316"+00	.172"-01	.585"-02
-0.4	.398"+00	.259"-01	.893"-02
-0.3	.501"+00	.380"-01	.134"-01
-0.2	.631"+00	.536"-01	.194"-01
-0.1	.794"+00	.716"-01	.269"-01
0.0	.100"+01	.890"-01	.354"-01
0.1	.126"+01	.102"+00	.432"-01
0.2	.158"+01	.106"+00	.487"-01
0.3	.200"+01	.102"+00	.508"-01
0.4	.251"+01	.930"-01	.494"-01
0.5	.316"+01	.816"-01	.457"-01
0.6	.398"+01	.696"-01	.407"-01
0.7	.501"+01	.583"-01	.352"-01
0.8	.631"+01	.481"-01	.297"-01
0.9	.794"+01	.392"-01	.247"-01
1.0	.100"+02	.317"-01	.203"-01
1.1	.126"+02	.255"-01	.165"-01
1.2	.158"+02	.204"-01	.133"-01
1.3	.200"+02	.163"-01	.107"-01
1.4	.251"+02	.129"-01	.856"-02
1.5	.316"+02	.102"-01	.681"-02
1.6	.398"+02	.804"-02	.540"-02
1.7	.501"+02	.628"-02	.425"-02
1.8	.631"+02	.486"-02	.333"-02
1.9	.794"+02	.371"-02	.258"-02
2.0	.100"+03	.277"-02	.198"-02

TAU = 1.0"-04

Y	LABDA	G1	G2
-2.0	.100"-01	.187"-04	.625"-05
-1.9	.126"-01	.297"-04	.990"-05
-1.8	.158"-01	.471"-04	.157"-04
-1.7	.200"-01	.746"-04	.249"-04
-1.6	.251"-01	.118"-03	.394"-04
-1.5	.316"-01	.187"-03	.625"-04
-1.4	.398"-01	.297"-03	.989"-04
-1.3	.501"-01	.470"-03	.157"-03
-1.2	.631"-01	.744"-03	.248"-03
-1.1	.794"-01	.118"-02	.393"-03
-1.0	.100"+00	.186"-02	.621"-03
-0.9	.126"+00	.293"-02	.980"-03
-0.8	.158"+00	.460"-02	.154"-02
-0.7	.200"+00	.720"-02	.242"-02
-0.6	.251"+00	.112"-01	.378"-02
-0.5	.316"+00	.172"-01	.585"-02
-0.4	.398"+00	.259"-01	.893"-02
-0.3	.501"+00	.380"-01	.134"-01
-0.2	.631"+00	.536"-01	.194"-01
-0.1	.794"+00	.716"-01	.269"-01
0.0	.100"+01	.890"-01	.354"-01
0.1	.126"+01	.101"+00	.432"-01
0.2	.158"+01	.106"+00	.487"-01
0.3	.200"+01	.102"+00	.507"-01
0.4	.251"+01	.928"-01	.494"-01
0.5	.316"+01	.814"-01	.456"-01
0.6	.398"+01	.694"-01	.406"-01
0.7	.501"+01	.580"-01	.350"-01
0.8	.631"+01	.477"-01	.296"-01
0.9	.794"+01	.387"-01	.245"-01
1.0	.100"+02	.311"-01	.201"-01
1.1	.126"+02	.248"-01	.162"-01
1.2	.158"+02	.195"-01	.130"-01
1.3	.200"+02	.152"-01	.103"-01
1.4	.251"+02	.116"-01	.802"-02
1.5	.316"+02	.871"-02	.617"-02
1.6	.398"+02	.635"-02	.466"-02
1.7	.501"+02	.448"-02	.343"-02
1.8	.631"+02	.303"-02	.245"-02
1.9	.794"+02	.197"-02	.170"-02
2.0	.100"+03	.122"-02	.113"-02

TAU = 1.0"-03

Y	LABDA	G1	G2
-2.0	.100"-01	.187"-04	.625"-05
-1.9	.126"-01	.297"-04	.990"-05
-1.8	.158"-01	.471"-04	.157"-04
-1.7	.200"-01	.746"-04	.249"-04
-1.6	.251"-01	.118"-03	.394"-04
-1.5	.316"-01	.187"-03	.625"-04
-1.4	.398"-01	.297"-03	.989"-04
-1.3	.501"-01	.470"-03	.157"-03
-1.2	.631"-01	.744"-03	.248"-03
-1.1	.794"-01	.118"-02	.393"-03
-1.0	.100"+00	.186"-02	.621"-03
-0.9	.126"+00	.293"-02	.980"-03
-0.8	.158"+00	.460"-02	.154"-02
-0.7	.200"+00	.720"-02	.242"-02
-0.6	.251"+00	.112"-01	.378"-02
-0.5	.316"+00	.172"-01	.585"-02
-0.4	.398"+00	.259"-01	.892"-02
-0.3	.501"+00	.379"-01	.133"-01
-0.2	.631"+00	.535"-01	.193"-01
-0.1	.794"+00	.714"-01	.269"-01
0.0	.100"+01	.886"-01	.352"-01
0.1	.126"+01	.101"+00	.430"-01
0.2	.158"+01	.105"+00	.484"-01
0.3	.200"+01	.101"+00	.503"-01
0.4	.251"+01	.912"-01	.488"-01
0.5	.316"+01	.794"-01	.448"-01
0.6	.398"+01	.670"-01	.396"-01
0.7	.501"+01	.550"-01	.338"-01
0.8	.631"+01	.441"-01	.281"-01
0.9	.794"+01	.345"-01	.228"-01
1.0	.100"+02	.263"-01	.180"-01
1.1	.126"+02	.193"-01	.138"-01
1.2	.158"+02	.137"-01	.103"-01
1.3	.200"+02	.933"-02	.745"-02
1.4	.251"+02	.606"-02	.518"-02
1.5	.316"+02	.377"-02	.347"-02
1.6	.398"+02	.224"-02	.223"-02
1.7	.501"+02	.128"-02	.138"-02
1.8	.631"+02	.704"-03	.829"-03
1.9	.794"+02	.376"-03	.480"-03
2.0	.100"+03	.195"-03	.270"-03

TAU = 1.0"-02

Y	LABDA	G1	G2
-2.0	.100"-01	.187"-04	.625"-05
-1.9	.126"-01	.297"-04	.990"-05
-1.8	.158"-01	.471"-04	.157"-04
-1.7	.200"-01	.746"-04	.249"-04
-1.6	.251"-01	.118"-03	.394"-04
-1.5	.316"-01	.187"-03	.625"-04
-1.4	.398"-01	.297"-03	.989"-04
-1.3	.501"-01	.470"-03	.157"-03
-1.2	.631"-01	.744"-03	.248"-03
-1.1	.794"-01	.118"-02	.393"-03
-1.0	.100"+00	.186"-02	.620"-03
-0.9	.126"+00	.293"-02	.979"-03
-0.8	.158"+00	.460"-02	.154"-02
-0.7	.200"+00	.718"-02	.242"-02
-0.6	.251"+00	.111"-01	.377"-02
-0.5	.316"+00	.170"-01	.582"-02
-0.4	.398"+00	.256"-01	.885"-02
-0.3	.501"+00	.373"-01	.132"-01
-0.2	.631"+00	.522"-01	.190"-01
-0.1	.794"+00	.689"-01	.261"-01
0.0	.100"+01	.842"-01	.339"-01
0.1	.126"+01	.941"-01	.407"-01
0.2	.158"+01	.957"-01	.450"-01
0.3	.200"+01	.891"-01	.458"-01
0.4	.251"+01	.773"-01	.431"-01
0.5	.316"+01	.632"-01	.380"-01
0.6	.398"+01	.489"-01	.317"-01
0.7	.501"+01	.359"-01	.250"-01
0.8	.631"+01	.248"-01	.188"-01
0.9	.794"+01	.162"-01	.134"-01
1.0	.100"+02	.100"-01	.909"-02
1.1	.126"+02	.588"-02	.588"-02
1.2	.158"+02	.326"-02	.362"-02
1.3	.200"+02	.172"-02	.214"-02
1.4	.251"+02	.868"-03	.121"-02
1.5	.316"+02	.417"-03	.657"-03
1.6	.398"+02	.191"-03	.344"-03
1.7	.501"+02	.832"-04	.174"-03
1.8	.631"+02	.346"-04	.848"-04
1.9	.794"+02	.137"-04	.401"-04
2.0	.100"+03	.516"-05	.184"-04

TAU = 1.0"-01

Y	LABDA	G1	G2
-2.0	.100"-01	.167"-04	.625"-05
-1.9	.126"-01	.297"-04	.990"-05
-1.8	.158"-01	.471"-04	.157"-04
-1.7	.200"-01	.746"-04	.249"-04
-1.6	.251"-01	.118"-03	.394"-04
-1.5	.316"-01	.187"-03	.624"-04
-1.4	.398"-01	.296"-03	.939"-04
-1.3	.501"-01	.469"-03	.156"-03
-1.2	.631"-01	.741"-03	.248"-03
-1.1	.794"-01	.117"-02	.391"-03
-1.0	.100"+00	.184"-02	.617"-03
-0.9	.126"+00	.289"-02	.971"-03
-0.8	.158"+00	.451"-02	.152"-02
-0.7	.200"+00	.698"-02	.237"-02
-0.6	.251"+00	.107"-01	.364"-02
-0.5	.316"+00	.159"-01	.552"-02
-0.4	.398"+00	.230"-01	.818"-02
-0.3	.501"+00	.318"-01	.117"-01
-0.2	.631"+00	.413"-01	.159"-01
-0.1	.794"+00	.492"-01	.202"-01
0.0	.100"+01	.529"-01	.236"-01
0.1	.126"+01	.506"-01	.251"-01
0.2	.158"+01	.430"-01	.240"-01
0.3	.200"+01	.325"-01	.207"-01
0.4	.251"+01	.221"-01	.162"-01
0.5	.316"+01	.136"-01	.116"-01
0.6	.398"+01	.762"-02	.759"-02
0.7	.501"+01	.392"-02	.463"-02
0.8	.631"+01	.186"-02	.263"-02
0.9	.794"+01	.820"-03	.141"-02
1.0	.100"+02	.338"-03	.715"-03
1.1	.126"+02	.131"-03	.346"-03
1.2	.158"+02	.478"-04	.161"-03
1.3	.200"+02	.167"-04	.722"-04
1.4	.251"+02	.557"-05	.316"-04
1.5	.316"+02	.180"-05	.135"-04
1.6	.398"+02	.561"-06	.567"-05
1.7	.501"+02	.171"-06	.235"-05
1.8	.631"+02	.508"-07	.961"-06
1.9	.794"+02	.148"-07	.391"-06
2.0	.100"+03	.426"-08	.158"-06

4. ALGOL 60 PROCEDURES

In this section we give the ALGOL 60 procedures for the computation of $H_\alpha(\lambda)$. The <u>real</u> <u>procedure</u> *halabda* computes this function with input parameters *a* (for α), *tau* (for τ) and *labda* (for λ). For $\lambda \geq .1$ *halabda* uses an <u>array</u> *g*[i,j], which contains function values of the function G(k); $i = 0,1,2,3,4,5$ corresponds to $\tau = 0, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}$, respectively, and j runs from 0 up to (and including) 64. These array values are read before calling *halabda*, and are computed by the <u>real</u> <u>procedure</u> *gbk*, which is also listed.

ACKNOWLEDGEMENT.

Thanks are due to Sjef Zimmerman for his collaboration in writing the Introduction and to GertJan Laan for his help in writing and testing the ALGOL 60 programs.


```
"REAL" "PROCEDURE" HALABDA(A, TAU, LABDA); "VALUE" A,TAU,LABDA;
       "REAL" A,TAU,LABDA;
       "IF" LABDA<.1 "THEN"
       HALABDA:=2*LABDA**(2*A)*(1/64-(A+2)*LABDA*LABDA
       *(9/2560+TAU/32-(A+3)*LABDA*LABDA
       *(19/40960+(27/3200+TAU/32)*TAU)))*
       ("IF" ABS(A-2)<.001 "THEN" 6 "ELSE" "IF" ABS(A-1)<.001
       "THEN" 2 "ELSE" 1.32934) "ELSE"
"BEGIN" "INTEGER" I,J;
        "REAL" X,Y,Z,T,AA,LL,SUM,TERM,H;
        I:= "IF" TAU = 0 "THEN" 0 "ELSE" ENTIER(6 + 0.43 * LN(TAU));
        SUM:=0;
        LL:=LABDA*LABDA;
        AA:=2*A+2;
        H:=.1171875;
        "FOR" J:=0, J+1 "WHILE" J<10 "OR"
              ( J<65 "AND" TERM/SUM>"-5) "DO"
        "BEGIN"
              X:=-1+J*H; Y:=EXP(-X); Z:=EXP(X-Y);
              TERM:=EXP( AA*(X-Y)-Z*Z/LL )*G[I,J];
              SUM:=SUM+TERM
        "END";

        HALABDA:=1.27324/LL/LL*H*SUM
"END" HALABDA;
```

```
"REAL" "PROCEDURE" GBK(K, T); "VALUE" K,T; "REAL" K,T;
"BEGIN" "REAL" X,B; "ARRAY" A,C[0:1000];


"REAL" "PROCEDURE" GBK1(K,T);"VALUE" K,T;"REAL" K,T;
"BEGIN" "REAL" B,KK,BB,BMON,BMEV,K2M,DMON,DMEV,S,TERM;
        "INTEGER" J,M,MLAST,P;

        KK:=K*K;
        B:=1+T*KK;
        BB:=B*B;
        S:=A[0]:=.5/(1+BB);

        DMEV:=ARCTAN(1);
        K2M:=1;
        C[0]:=DMEV*S*S;

        "FOR" M:=2,M+2 "WHILE" ABS(TERM)>"-5 "AND" M<1000 "DO"
        "BEGIN" A[M-1]:=-(M-.5)*A[M-2]/M/(M*M+BB);
                A[M]:=-(M+.5)*A[M-1]/(M+1)/((M+1)*(M+1)+BB);
                P:=M/2-1;
                BMEV:=BMON:=0;
                "FOR" J:=0 "STEP" 1 "UNTIL" P "DO"
                "BEGIN" BMON:=BMON+2*A[J]*A[M-1-J];
                        BMEV:=BMEV+2*A[J]*A[M-J];
                "END";

                BMEV:=BMEV+A[P+1]*A[P+1];
                DMON:=(M-.5)*DMEV/M;
                DMEV:=(M+.5)*DMON/(M+1);
                C[M-1]:=DMON*BMON;
                C[M]:=DMEV*BMEV;
                MLAST:=M;
                K2M:=KK*K2M*KK;
                TERM:=DMEV*BMEV*K2M;
        "END";
        M:=M;

        "COMMENT" HORNER;
        S:=C[MLAST];
        "FOR" J:=MLAST-1 "STEP" -1 "UNTIL" 0 "DO"
              S:=S*KK+C[J];
        GBK1:=KK*S

"END" GBK1;


"REAL" "PROCEDURE" FB(X,B);"VALUE" X,B; "REAL" X,B;
"BEGIN" "ARRAY" A[0:1000];

        "REAL" "PROCEDURE" FB1(X,B); "VALUE" X,B; "REAL" X,B;
```

```
"BEGIN" "REAL" XX,T,S,X2M,BB;
        "INTEGER" M;
        BB:=B*B;
        S:=A[0]:=.5/(1+BB);
        XX:=X*X; X2M:=1;
        "FOR" M:=1, M+1 "WHILE" ABS(T/S)>"-5
        "DO" "BEGIN" X2M:=XX*X2M;
             A[M]:=-(M+.5)*A[M-1]/(M+1)/((M+1)*(M+1)+BB);
             T:=A[M]*X2M;
             S:=S+T
        "END";
        FB1:=(X*S)**2
"END" FB1;

"REAL" "PROCEDURE" FB2(X,B);"VALUE" X,B; "REAL" X,B;
"BEGIN" "REAL" P,Y,BB;
        "INTEGER" N,NN;
        P:=5.76/X; Y:=(3*P)**.6666667; BB:=B*B;
        N:=ENTIER(X*(1+Y*(.5+Y*(.008333-.000357*Y))));

        "BEGIN" "ARRAY" J[0:N];
                BESSJ(X,N,J); P:=0;
                "FOR" N:=N,N-1 "WHILE" N>0 "DO"
                "BEGIN" NN:=N*N;
                        P:=P+NN/(NN+BB)*J[N]**2
                "END"
        "END";
        FB2:=(2*P/X)**2
"END" FB2;

"REAL" "PROCEDURE" FB3(X,B); "VALUE" X,B; "REAL" X,B;
"BEGIN" "REAL" P,Q,XX,PP,QQ,JJ;
        XX:=1/X/X;
        P:=(((62.16631*XX-5.623960)*XX+1.016286)*XX-.507813)*XX+
             1;
        Q:=-((((404.773893*XX-27.798258)*XX+3.455373)*
                XX-.981771)*XX+1)*.625/X;
        PP:=P*P;
        QQ:=Q*Q;
        JJ:=(11.591953*(PP+QQ)+SIN(X+X)*(PP-QQ)
             +2*COS(X+X)*P*Q)/X;
        FB3:=((1-JJ/11.548739)/X)**2
"END" FB3;

"PROCEDURE" BESSJ(X,N,J);"VALUE" X,N;"REAL" X;
 "INTEGER" N; "ARRAY" J;
 "IF" X=0 "THEN"
 "BEGIN" J[0]:=1;
         "FOR" N:=N "STEP" -1 "UNTIL" 1 "DO" J[N]:=0
 "END" "ELSE"
 "BEGIN" "REAL" X2,R,S,P,Y; "INTEGER" L,M,NU,SIGNX;
```

```
          SIGNX:=SIGN(X); X:=ABS(X);
          P:=11.51/X; Y:=(3*P)**.66666667;
          NU:=(1 + ENTIER(X*(((-Y/2800+1/120)*Y+.5)*Y+1)))//2*2;
          R:=S:=0; X2:=2/X; L:=0;
          "FOR" M:=NU "STEP" -1 "UNTIL" 1 "DO"
          "BEGIN" R:=1/(X2*M-R);
                  L:=2-L; S:=R*(L+S);

                          "IF" M<=N "THEN" J[M]:=R
                 "END";
                 J[0]:=R:=1/(1+S);
                 "FOR" M:=1 "STEP" 1 "UNTIL" N "DO"
                 J[M]:=R:=R*J[M];
                 "IF" SIGNX<0 "THEN"
                 "FOR" M:=1 "STEP" 2 "UNTIL" N "DO"
                 J[M]:=-J[M]
        "END" BESSELJ;


        FB:="IF" X<6.5 "THEN" FB1(X,B) "ELSE"
                      "IF" B>1 "THEN" FB2(X,B) "ELSE" FB3(X,B)

"END" FB;

"REAL" "PROCEDURE" TRAP(A,B,X,FX,N);"VALUE" A,B,N; "REAL" A,B,X,FX;
"INTEGER" N;
"BEGIN" "REAL" E,H,P,Q,V; "INTEGER"I;
        H:=B-A; P:=0; E:=.5*10**(-N);
        "FOR" X:=A,B "DO" P:=P+FX; P:=H*P/2; I:=0;
        "FOR" H:=H, H/2 "WHILE" V>E "AND" I<10 "DO"
        "BEGIN" Q:=0; "FOR" X:=A+H/2, X+H "WHILE" X<B "DO" Q:=Q+FX;
                Q:=Q*H; V:=ABS((P-Q)/P); P:=(P+Q)/2; I:=I+1
        "END";
        TRAP:=P
"END" TRAP;


B:=1+T*K*K;

GBK:="IF" K<=6.5 "THEN" GBK1(K,T)
                 "ELSE" TRAP(0,1.570796,X,FB(K*COS(X),B),3)

"END" GBK;
```

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